J. Fluid Mech. (1969), *vol.* 37, *part* 3, *pp.* 497–512
Printed in Great Britain

Self-similar piston problems with radiative heat transfer

By J. B. HELLIWELL
School of Mathematics, The University of Bradford

(Received 6 September 1968)

A differential approximation for the equations of radiative transfer in a grey gas is applied in a study of the effects of thermal radiation upon the classical problem of the compressive action of a plane, cylindrical or spherical piston. The ambient gas ahead of the precursor shock wave is supposed cool and the shock wave transparent, whilst the piston is taken to be neither an emittor nor reflector of radiative energy. It is shown that self-similar flow patterns may arise if the ambient density and piston speed are both non-uniform with variations linked to the absorption coefficient which is assumed to be density and temperature dependent. Detailed flow patterns are obtained in the case of general opacity and also in the transparent limit from which it is deduced that under certain conditions the approximation provided by the latter may be rather dubious.

1. Introduction

The problems of radiative energy transfer in fluids have received increasing attention in recent years as a consequence of the increasing speeds of bodies through the atmosphere and the very high temperatures attained by gases in motion. Effects of radiation are of significance in the fields of nuclear power and space research, for instance.

However, the phenomena associated with heat transfer in a radiating fluid are extremely complex whether the fluid is at rest or in motion, steady or unsteady. Whilst the underlying physical concepts have been understood for a considerable time, the mathematical formulation of a model to represent the behaviour of a radiating gas leads to a coupled system of differential, integral and integrodifferential equations. The difficulty in solving such systems is very considerable and progress in the analysis of radiative gasdynamics has been slow. In addition to this inherent barrier to progress added complications are generated by the essential non-linearity of the equations and large number of parameters involved, together with the fundamental dependence of the radiative flux upon the geometrical configuration and structure of the boundaries of the physical problem.

The consequence of these complexities has been to stimulate a search for approximate formulation of the equations of radiative transfer together with, perhaps, an over-emphasis upon plane problems with boundaries at infinity. Moreover, in the majority of studies the conditions are either stationary or steady, as for example in the case of radiative shock structure analyses. Nevertheless, it is important that an effort should also be directed towards other aspects of radiative gasdynamics and the work reported here has been undertaken with this in mind. An important facet of this investigation is the use of a more general approximation, retaining finite opacity, for the equations of radiative transfer in contrast with the transparent or Rosseland limits corresponding respectively to optically thin and thick gases. A direct assessment of the transparent approximation is thus made in this paper and under some circumstances it has been shown to be inadequate. In addition, the radiative boundary conditions have been imposed at finite, rather than infinite, points, with corresponding enhancement in our understanding of their effects. Finally the problems studied are unsteady in both planar and non-planar geometries so that from the patterns of behaviour which are established one may obtain an estimate of the effects of curvilinear configurations.

The various forms of approximation which have been used to simplify the governing equations of radiative transfer lead to a system of equations which are entirely differential. One such type of approximation is indeed termed the differential approximation and the particular form which is employed here is that developed in earlier work by Helliwell (1966) in which, as a result of using a truncated series expansion for the radiative intensity and a moment-generating method, a system of differential equations has been established for the local backward and forward fluxes of radiation, valid for three-dimensional configurations and general opacity of the gas. It is the purpose of this paper to apply these equations in a study of the effects of radiative heat transfer upon the classical problem of a plane, cylindrically symmetric or spherically symmetric piston, thrust by compressive action into a gas to generate a strong precursor shock wave. In the non-planar cases the motion of the piston is taken to be outwards from the central axis or point of symmetry so that the shock wave is explosive, rather than implosive. The problem in the plane case has been examined previously by Wang (1964) who employed a modification of the Schuster–Schwarzschild differential approximation for the equations of radiative flux. In addition to using an improved form of approximation the present work also extends the investigation to flows with non-planar geometry.

In an endeavour to obtain a solution to the problem sufficiently detailed for precise conclusions to be drawn as to the major radiative effects, a number of simplifications are made concerning the gas and piston properties. Specifically, a perfect grey gas in local thermodynamic equilibrium has been chosen so that the equation of state is not complex, the radiative effects are presumed frequency independent and it is meaningful to refer to a gas temperature. Furthermore, the radiative pressure tensor and energy density are taken to be negligibly small, an assumption which is known to be reasonable provided that the temperature is not extremely high nor the gas very tenuous. Finally it is assumed that the ambient gas ahead of the shock wave is cool and the shock wave itself is transparent to radiation, with the piston also cool and non-reflecting, so that no radiative flux enters the gas between the shock and piston across either shock or piston boundaries.

Now, in the absence of radiation, the general aspects of piston problems are well known. The early work of Taylor (1946) upon the subject laid the foundations for later studies and in his book Sedov (1959) has systematized the analysis appropriate to the class of self-similar solutions. When the effects of radiation are not negligible the problems are shown to form an extension of this class provided that there holds a simple relationship between certain exponents in the expressions for the piston speed, the ambient density and the absorption coefficient, all of which may be supposed non-uniform. The particular forms chosen for these are stated explicitly as equations (11), (12) and (10) respectively. The latter is a consequence of the data concerning optical properties. To justify the introduction of the form (11) it may be observed that, in non-radiative hypersonic flow theory with slender bodies possessing power law profiles, the flow in the shock layer is given by the solution of analogous unsteady piston problems with the stated piston speed. It is thus natural to take this same relationship for the corresponding radiative piston problems (following Wang). With regard to equation (12), whilst the form is essential for the generation of self-similar flow patterns, yet it is also of interest to note that a power law spatial variation of ambient density may be relevant in astrophysics, where radiative transfer is of considerable importance. The properties of such self-similar solutions are examined and detailed flow patterns are obtained for various geometries, specific heat ratios and absorption coefficients. The effects of thermal radiation are deduced in both the cases of general radiative transfer and the transparent limit from which it is noted that the latter may not provide an assessment even qualitatively accurate.

2. Governing equations and self-similar formulation

The fundamental equations governing the one-dimensional motion of an inviscid perfect gas in which the effects of radiative flux may be significant are statements relating to conservation of mass, momentum and energy, and can be written in the form

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial r}(\rho v)+(\nu-1)\frac{\rho v}{r} = 0, \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}+\frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \tag{2}$$

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial r}\right)e+p\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial r}\right)\frac{1}{\rho}+\frac{1}{\rho}\left(\frac{\partial}{\partial r}+\frac{\nu-1}{r}\right)q = 0. \tag{3}$$

Here v is the speed of the gas, p the pressure, ρ the density, e the specific internal energy, t the time and r is the single spatial co-ordinate being either axial in flows with planar geometry or radial in cylindrically and spherically symmetric flows. The constant ν takes the value 1, 2 or 3 according to the dimensions of the respective geometry. In addition q denotes the magnitude of the flux of thermal radiation along the co-ordinate direction and is separated into its forward and backward components, $q_{\mp}$, respectively, so that

$$q = q_{-}-q_{+}. \tag{4}$$

Furthermore, the equation of state of the gas is taken to be of the form

$$p = \mathscr{R}\rho T, \tag{5}$$

where $\mathscr{R}$ is the gas constant and T is the temperature, together with

$$e = \frac{p}{(\gamma-1)\rho}, \tag{6}$$

where γ is the ratio of the specific heats. Finally the equations under differential approximation for the variation in the radiative flux components may be written, following Helliwell, as follows:

general opacity

$$\left(\frac{\partial}{\partial r}+\frac{\nu-1}{r}\right)(q_- - q_+) = 4\pi kB - 2k(q_- + q_+), \tag{7}$$

$$\frac{\partial}{\partial r}(q_- + q_+) = -\tfrac{3}{2}k(q_- - q_+); \tag{8}$$

transparent limit

$$\frac{\partial}{\partial r}q + \frac{(\nu-1)}{r}q = 4\pi kB; \tag{9}$$

where B denotes Planck's radiation function and is given by $B = \sigma T^4/\pi$, σ is Stefan's constant and k is the local volumetric absorption coefficient.

Whilst in reality the absorption coefficient is a complex function of the properties of the gas it is assumed for the present analysis that its true character may be approximated by a simple relationship involving solely the density and temperature, of the form

$$k = K\rho^\alpha T^\beta. \tag{10}$$

Representative values of grey absorption coefficients have been presented by Armstrong *et al.* (1961) and from these it can be shown that for a specific range of temperature the above expression has reasonable validity, the values of K, α and β varying, however, with the corresponding range. Thus, for densities of the order of the ambient value at mean sea level, viz. $\rho \simeq 1{\cdot}29\times 10^{-3}\,\mathrm{g/cm^3}$, one finds

$$\alpha = 1, \quad \beta = 5, \quad K = 2\times 10^{-18} \quad \text{for} \quad T < 2\times 10^4;$$

$$\alpha = 2, \quad \beta = -1, \quad K = 7\times 10^{11} \quad \text{for} \quad 9\times 10^4 < T < 2\times 10^5.$$

Now, as in the non-radiative case, (1)–(3) contain no dimensional constants. However, when these are combined with the remaining equations (4)–(8) and (10) relevant to a gas of general opacity there arise two constants, of independent dimensions, which may be chosen to be the following:

$$K\sigma/\mathscr{R}^{\beta+4} \quad \text{with dimensions} \quad M^{1-\alpha} \quad L^{3\alpha-2\beta-9} \quad T^{2\beta+5},$$

$$K/\mathscr{R}^{\beta} \quad \text{with dimensions} \quad M^{-\alpha} \quad L^{3\alpha-2\beta-1} \quad T^{2\beta},$$

where M, L, T are quantities possessing dimensions of mass, length and time, respectively. In the transparent limit it can be shown that only the first constant occurs. Thus, since it is known that a self-similar flow pattern may exist and the governing equations reduce to ordinary differential form provided that exactly two independent dimensional constants arise in the complete formulation, including boundary conditions, it follows that in self-similar radiative piston problems no additional constants with independent dimensions may present themselves in the boundary conditions.

Hence we consider the non-uniform motion of a piston thrust with speed

$$U = U_0 t^n \quad (n > -1) \tag{11}$$

into a gas at rest in which the density varies with position according to the law

$$\rho = \rho_1 = \rho_0 r^{-\omega} \quad (\omega > 0), \tag{12}$$

where n and ω are arbitrary constants. It is known that ahead of the piston must run a shock wave which in a self-similar flow pattern must be strong in order that the ambient pressure ahead may be neglected compared with that behind, otherwise three constants of independent dimensions arise in the boundary conditions so far formulated. The two constants which remain are

$$U_0 \quad \text{with dimensions} \quad LT^{-(n+1)},$$

$$\rho_0 \quad \text{with dimensions} \quad ML^{\omega-3}.$$

With regard to the radiative boundary conditions, it is assumed that the piston is such that it neither emits nor reflects radiation, and thus may be said to be cool and black. In addition the gas ahead of the shock wave is also supposed cool so that it does not emit radiative energy, and thus no radiative flux passes into the gas behind the shock wave from upstream. Furthermore, the transition region within the shock wave itself is taken to be transparent to the radiative flux, so that the jump equations across the front reduce to those of a strong non-radiative shock passing into a gas at rest. Hence

$$v_2 = 2c/(\gamma+1), \tag{13}$$

$$\rho_2 = \{(\gamma+1)/(\gamma-1)\}\rho_1, \tag{14}$$

$$p_2 = \{2/(\gamma+1)\}\rho_1 c^2, \tag{15}$$

where c is the shock speed and the suffixes 1 and 2 denote conditions upstream and downstream respectively. Thus for the region of disturbed gas between the piston and precursor shock wave, (12)–(15) together with

$$q_+ = 0 \tag{16}$$

provide the boundary conditions immediately downstream of the shock, whilst (11) together with

$$q_- = 0 \tag{17}$$

yield the corresponding conditions at the piston face. Clearly among these conditions there exist a further two independent dimensional constants, so that a self-similar radiative piston problem may be formulated if the dimensions of these constants are consistent with those arising from the governing equations. This is so provided that

$$\omega = \frac{5}{5\alpha+2\beta}, \quad n = \frac{-\omega}{\omega+5}, \tag{18}$$

as determined previously by Wang (1964) for the plane case. Rather less restrictive conditions are necessary in the transparent limit.

For self-similar problems, the independent dimensional constants U_0, ρ_0 may then be taken as basic and a dimensionless similarity variable, λ, can be introduced, defined by

$$\lambda = \left(\frac{\delta\lambda^*}{U_0}\right) r t^{-\delta}, \tag{19}$$

where $\delta = n+1$ and the parameter λ^* is inserted so that immediately behind the shock wave one may choose $\lambda = 1$. It then follows that $\lambda = \lambda^*$ at the piston face. The field variables describing the flow pattern are then able to be written in terms of dimensionless functions of λ, such that

$$v = \frac{rV(\lambda)}{t}, \quad \rho = \frac{\rho_0 R(\lambda)}{r^\omega}, \quad p = \frac{\rho_0 P(\lambda)}{r^{\omega-2}t^2}, \quad q_\pm = \frac{\rho_0 Q_\pm(\lambda)}{r^{\omega-3}t^3}. \tag{20}$$

It is also convenient to define a dimensionless acoustic speed in terms of the variable z, where

$$\frac{\gamma p}{\rho} = \frac{\gamma P}{R}\left(\frac{r}{t}\right)^2 = z\left(\frac{r}{t}\right)^2. \tag{21}$$

In terms of these new variables it is now a straightforward but tedious matter to obtain the appropriate forms of the governing equations and associated boundary conditions.

For a gas of general opacity one finds:

$$\lambda\frac{dV}{d\lambda} = \frac{V(V-1)(\delta-V)-(\kappa-\nu V)+f}{(\delta-V)^2-z}, \tag{22}$$

$$\frac{\lambda}{R}\frac{dR}{d\lambda} = \frac{(\omega-\nu)V\{z-(\delta-V)^2\}+(\delta-V)V(V-1)-z(\kappa-\nu V)+f}{(\delta-V)\{(\delta-V)^2-z\}}, \tag{23}$$

$$\lambda\frac{dz}{d\lambda} = \frac{z(\delta-V)^2[\{2+(\gamma-1)\nu\}V-2]+(\gamma-1)zV(V-1)(\delta-V) - z^2\{2V+(\gamma-1)\kappa-2\}+f\{\gamma(\delta-V)^2-z\}}{(\delta-V)\{(\delta-V)^2-z\}}, \tag{24}$$

$$2\lambda\frac{dQ_-}{d\lambda} = \{(2\omega-\nu-5)-\tfrac{7}{4}K_2\lambda^{2\beta/\delta}z^\beta R^\alpha\}Q_- + \{(\nu-1)-\tfrac{1}{4}K_2\lambda^{2\beta/\delta}z^\beta R^\alpha\}Q_+ + K_1\lambda^{(2\beta+5)/\delta}z^{\beta+4}R^\alpha, \tag{25}$$

$$2\lambda\frac{dQ_+}{d\lambda} = \{(\nu-1)+\tfrac{1}{4}K_2\lambda^{2\beta/\delta}z^\beta R^\alpha\}Q_- + \{(2\omega-\nu-5)+\tfrac{7}{4}K_2\lambda^{2\beta/\delta}z^\beta R^\alpha\}Q_+ - K_1\lambda^{(2\beta+5)/\delta}z^{\beta+4}R^\alpha, \tag{26}$$

where

$$f = f(z,R,Q_-,Q_+,\lambda) = \frac{(\gamma-1)z^\beta R^{\alpha-1}\lambda^{2\beta/\delta}\{K_1 z^4\lambda^{5/\delta}-K_2(Q_-+Q_+)\}}{(\delta-V)^2-z}, \tag{27}$$

$$\gamma\kappa = (\omega-2)\delta+2,$$

$$K_1 = \frac{4K\sigma\rho_0^{\alpha-1}}{(\gamma\mathscr{R})^{\beta+4}}\left(\frac{U_0}{\delta\lambda^*}\right)^{(2\beta+5)/\delta}, \quad K_2 = \frac{2K\rho_0^\alpha}{(\gamma\mathscr{R})^\beta}\left(\frac{U_0}{\delta\lambda^*}\right)^{2\beta/\delta},$$

subject to the boundary conditions

$$V = \frac{2\delta}{\gamma+1}, \quad R = \frac{\gamma+1}{\gamma-1}, \quad z = \frac{2\gamma(\gamma-1)\delta^2}{(\gamma+1)^2}, \quad Q_+ = 0 \quad \text{at} \quad \lambda = 1; \tag{28}$$

$$V = \delta, \quad Q_- = 0 \quad \text{at} \quad \lambda = \lambda^*. \tag{29}$$

In the transparent limit the appropriate equations may be immediately deduced from the general set by putting $K_2 = 0$. The equations for V, R and z then become independent of the radiative flux components, $Q_{\pm}$, and may be solved separately. In terms of this solution the variation of Q with λ is then obtained by means of a single quadrature, and one finds

$$Q = \int_{\lambda^*}^{\lambda} \left(\frac{\eta}{\lambda}\right)^{2+\nu-\omega} g(\eta)\frac{d\eta}{\eta} - \int_{\lambda}^{1} \left(\frac{\eta}{\lambda}\right)^{2+\nu-\omega} g(\eta)\frac{d\eta}{\eta} + \left(\frac{1-\lambda^{*\nu-1}}{1+\lambda^{*\nu-1}}\right)\int_{\lambda^*}^{1} \left(\frac{\eta}{\lambda}\right)^{2+\nu-\omega} g(\eta)\frac{d\eta}{\eta}, \tag{30}$$

where $$g(\eta) = \tfrac{1}{2}K_1 z^{\beta+4} R^{\alpha} \eta^{(2\beta+5)/\delta}$$

and z, R are regarded as functions of η.

Finally it is necessary to obtain from (19) and (20) the relationship between the similarity solution and the physical variables. The most convenient form of this is the following:

$$\frac{v}{v_s} = \frac{(\gamma+1)V\lambda}{2\delta}, \quad \frac{\rho}{\rho_s} = \left(\frac{\gamma-1}{\gamma+1}\right)R\lambda^{-\omega}, \quad \frac{p}{p_s} = \frac{(\gamma+1)zR\lambda^{2-\omega}}{2\gamma\delta^2},$$
$$\frac{T}{T_s} = \frac{(\gamma+1)^2 z\lambda^2}{2\gamma(\gamma-1)\delta^2}, \quad \frac{q}{q_s} = \frac{(Q_- - Q_+)\lambda^{3-\omega}}{Q_{s-}}, \tag{31}$$

where $\lambda = r/r_s$ and the suffix s denotes values immediately behind the shock front. The latter, apart from Q_{s-} which is determined from the solution itself, are given immediately by (12)–(15) and (19).

It is observed that in radiative flows the governing equations and boundary conditions (22)–(29) comprise a two-point boundary value problem. Furthermore, the system of ordinary differential equations is non-autonomous. Hence the existence of solutions for arbitrary α, β is difficult to ascertain with precision, and when they exist the details can only be obtained by extensive computation. Thus in order to gain an insight into the probable forms of solution and to lay a foundation upon which to build the effects of radiation, a study is first made of the simpler autonomous non-radiative case for arbitrary piston speed and ambient density.

3. Non-radiative piston problems

The non-radiative problem is governed by (22)–(24) in which $K_1 = 0 = K_2$, with the boundary conditions upon $Q_{\pm}$ omitted from the sets (28) and (29). Thus the solution may be determined in the (V, z)-plane from the equation obtained by eliminating λ from the pair (22) and (24). This is

$$\frac{dz}{dV} = \frac{z[(\delta-V)^2\{[2+(\gamma-1)\nu]V-2\}+(\gamma-1)V(\delta-V)(V-1) - z\{2V+(\gamma-1)\kappa-2\}]}{(\delta-V)[V(V-1)(\delta-V)-(\kappa-\nu V)z]}. \tag{32}$$

It should be noted that in the region behind the shock wave, since the flow there is subsonic, one has

$$z-(\delta-V)^2 > 0,$$

and the domain of interest in the (V, z)-plane is

$$2\delta/(\gamma+1) < V < \delta, \quad z > 0.$$

It is also recalled that the appropriate values of ω and δ in connexion with radiative problems are related primarily by equations (18) to the radiative parameters α, β which the opacity data of Armstrong *et al.* (1961) indicate are such that $1 \leqslant \alpha \leqslant 2$, $-5 \leqslant \beta \leqslant 7$. By and large it is reasonable to take

$$\alpha = 1, \quad 0 \leqslant \beta \leqslant 7 \rightarrow \tfrac{5}{19} \leqslant \omega \leqslant 1, \quad \tfrac{5}{6} \leqslant \delta \leqslant 1 \quad \text{for} \quad T < 10^{4.5}\,\text{degK},$$

$$\alpha = 2, \quad -5 \leqslant \beta \leqslant 0 \rightarrow \tfrac{1}{2} \leqslant \omega < \infty, \quad 0 < \delta \leqslant \tfrac{10}{11} \quad \text{for} \quad T > 10^{4.5}\,\text{degK}.$$

Thus the various cases may be investigated together by working in terms of the single parameter δ, which extends over the range $0 < \delta < 1$. However, differences in flow pattern arise according to the magnitude of γ for which we shall take two alternative values; either $\gamma = \frac{7}{5}$, which gives rise to certain special features, or $\gamma = \frac{5}{3}$, the results for which are typical of those for other values, $1 < \gamma < 2$. In what follows details of the solution are presented for $\gamma = \frac{7}{5}$ and, where appropriate, an indication is given of the corresponding more typical behaviour associated with $\gamma = \frac{5}{3}$.

For $\gamma = \frac{7}{5}$ the singularities of the differential equation (32) are as follows:

(i) $V = 0, z = 0$, a node;

(ii) $V = \delta, z = 0$, a saddle;

(iii) $V = 1, z = 0$, a node;

(iv) $V = 5/(5+\nu)$, $z = \nu/(5+\nu)^2$, a node or saddle depending upon the values of ν, δ;

(v) if $\nu = 1$, $V = 5\delta/4$, $z = \delta^2/16$, a node or saddle depending upon the value of δ. If $\nu = 2$ or 3, in place of (v) two singularities (v) and (vi) arise for δ sufficiently large both of which lie on the acoustic locus $z = (\delta - V)^2$;

(vii) $V = 5(1-\delta)/\nu$, $z = \infty$, a saddle.

In addition the integral curves have zero slopes on

$$z = 0, z = \{V - \tfrac{1}{5}(2+3\delta)\}^2 - \tfrac{1}{25}\{(9\delta-4)(\delta-1)\},$$

and infinite slopes on

$$z = \{V(V-1)(V-\delta)\}/\{V-5(1-\delta)\}.$$

Also, as (23)–(25) show, on these curves λ is an extremum at points of intersection with the acoustic locus, except at singularities. Thus, in general, for physically meaningful self-similar solutions to occur, integral curves must exist which extend from the shock point $V = 5\delta/6$, $z = 7\delta^2/36$ at $\lambda = 1$ to end on the piston path $V = \delta$ where $\lambda = \lambda^*(<1)$ without *en route* crossing the acoustic locus except possibly at a singularity. It is clear that in at least a segment of $5\delta/6 \leqslant V \leqslant \delta$ it is necessary that $dV/d\lambda < 0$. Hence from (22) since in this range $z - (V-\delta)^2 > 0$ and $V(V-1)(V-\delta) > 0$ and $z > 0$, it follows that solutions do not exist if $\delta \leqslant 5/(\nu+5)$. Therefore it appears that no solution exists if the singularity (iv) lies to the right of the piston path $V = \delta$. As the value of δ increases, a study of the integral curves shows that solutions cannot exist until a value is attained such that the shock point lies to the right of the unique integral curve which passes through the singularities (ii), (iv) and (vii). However, it is not possible to obtain an explicit inequality for δ from this condition since an analytic form is unobtainable for this curve. Nevertheless, a slightly weaker condition can be determined precisely, for a solution always exists if the shock point lies to the right and above the locus of infinite slopes of the integral curves. Hence solutions certainly exist for $\delta \geqslant 48/(47+7\nu)$. The integral curves appropriate to spherical pistons are shown in figure 1 for a value of δ when a solution exists even though this inequality is just not satisfied. For all realistic patterns, as that illustrated here, the solution curve ends upon the piston face at a regular point of (32) so that conditions throughout the field of flow are finite everywhere.

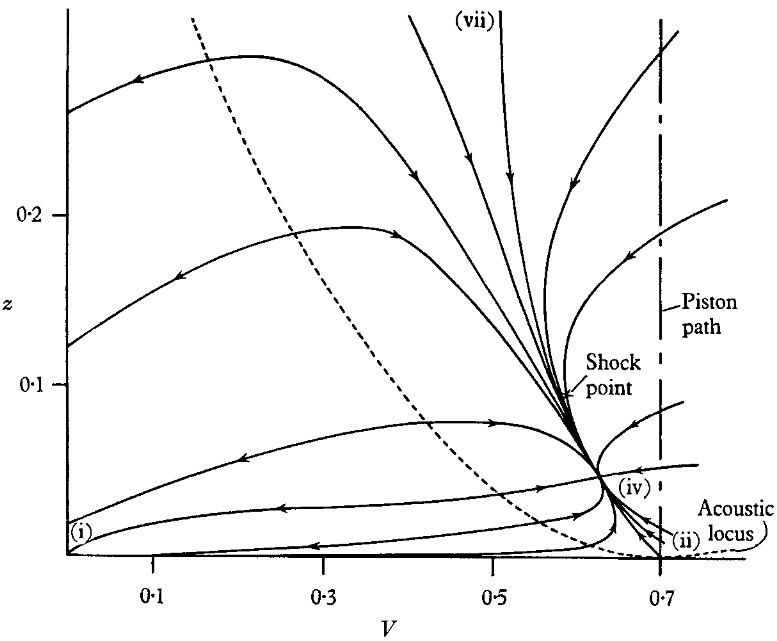

FIGURE 1. Integral curves. Non-radiative case: $\nu = 3$, $\gamma = \frac{7}{5}$, $\delta = \frac{7}{10}$. The arrows indicate the sense of increasing λ.

In the case when $\gamma = \frac{5}{3}$ the locations of the singularities (i), (ii) and (iii) of the governing equation (32) remain the same as for $\gamma = \frac{7}{5}$, but (ii) is now not simple. The singularities (iv), (v), (vi) and (vii) arise, as before, with the addition of

(viii) $V = \delta$, $z = \infty$, a saddle.

The analysis of the integral curves for varying δ, carried out as for the previous case, then shows that solutions cannot exist if $\delta \leqslant 21/(21+5\nu)$ and certainly do exist for $\delta \geqslant 32/(31+5\nu)$. In figure 2 is shown the system of integral curves for the planar case when $\nu = 1$ and the limiting value $\delta = \frac{8}{9}$ is chosen so that the shock point lies on the locus of infinite slopes. The general result that all solution curves enter the singularity (ii) is clearly portrayed here, so that at the piston face singular values occur for certain of the flow variables. Indeed it can be shown analytically that as the solution curve approaches (ii) the velocity and pressure remain finite but the temperature tends to zero whilst the density becomes unbounded.

The results of this section may be summarized as follows. For arbitrary γ in the range $1 < \gamma < 2$ solutions do not exist if $\delta \leqslant 7/(7+\gamma\nu)$. On the other hand, solutions do exist if $\delta \geqslant 8(1+\nu)/\{9+(7+2\nu)\gamma\}$. No definite statement is possible for values of δ between these limits, apart from the fact that solutions certainly continue to exist for values of δ a little below the second of these limits. Detailed computation has been made of the solution in the cases when $\alpha = 1$, $\beta = 5$ ($\delta = \frac{15}{16}$), and $\alpha = 2$, $\beta = -1$ ($\delta = \frac{8}{9}$) for all ν and both $\gamma = \frac{7}{5}$ and $\frac{5}{3}$. Apart from the density and temperature near the piston face, the results are essentially unchanged as the value of γ varies, and thus complete flow patterns are presented in figures 3–6 only for the plane and spherical pistons with $\gamma = \frac{7}{5}$. Similar calculations for $\alpha = 2$, $\beta = -2$ ($\delta = \frac{6}{7}$) show little difference apart from the non-existence of the solution when $\nu = 1$.

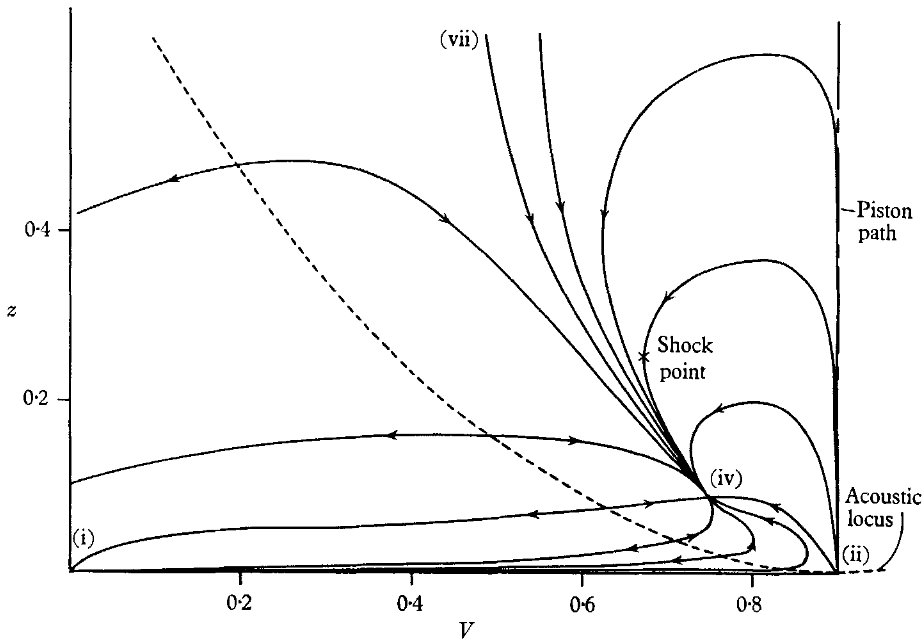

FIGURE 2. Integral curves. Non-radiative case: $\nu = 1, \gamma = \frac{5}{3}, \delta = \frac{8}{9}$. The arrows indicate the sense of increasing λ.

4. Radiative piston problems

The solution of the full system of radiative equations (22)–(27) under the boundary conditions (28) and (29) has been carried out upon a digital computer using the Runge–Kutta–Merson technique for the numerical integration. Despite the fact that an alternative procedure involving the quadrature (30) could have been employed in the transparent limit, for computational purposes it was found more convenient to use essentially the same process for all the radiative calculations.

In the transparent limit, as previously remarked, (22)–(24) for V, R and z become independent of $Q_{\pm}$ whilst (25) and (26) are two linear differential equations for $Q_{\pm}$. The boundary conditions (28) and (29) are such that values of V, R, Q_+ and z are given at $\lambda = 1$ whilst Q_- is unknown there. Hence by computing two solutions for arbitrarily specified Q_- at $\lambda = 1$, for each of which the forms of V, R and z are identical, it is a simple matter to combine these linearly so that Q_- is zero at $V = \delta$ and thereby to derive the solution which fits all the boundary

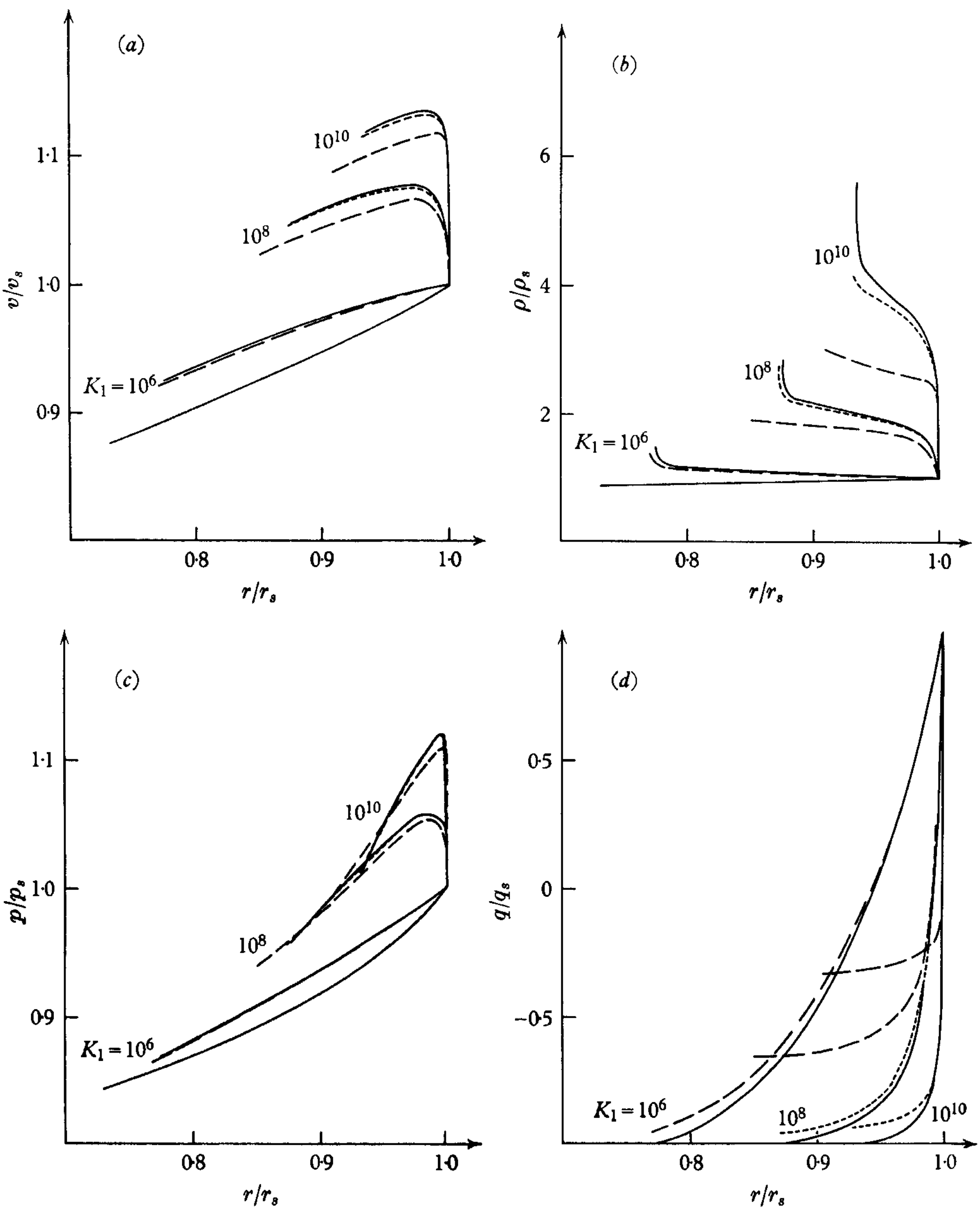

FIGURE 3. (*a*) Velocity distribution. (*b*) Density distribution. (*c*) Pressure distribution. (*d*) Radiative flux distribution. ———, non-radiative; ———, transparent radiative; general radiative: — —, $\rho_0 = 1{\cdot}29 \times 10^{-4}$; - - -, $\rho_0 = 1{\cdot}29 \times 10^{-5}$. Flow pattern: $\nu = 1$, $\gamma = \frac{7}{5}$, $\alpha = 1, \beta = 5$.

conditions. For the case of general opacity such a simple technique is not available. However, it is reasonable to make an estimate of the value of Q_- at $\lambda = 1$ by using the transparent solution, and employing this for a first calculation and a slightly different value for a second, neither of which satisfy $Q_- = 0$ at $V = \delta$, it is possible to design an iterative process which ultimately converges to the required solution.

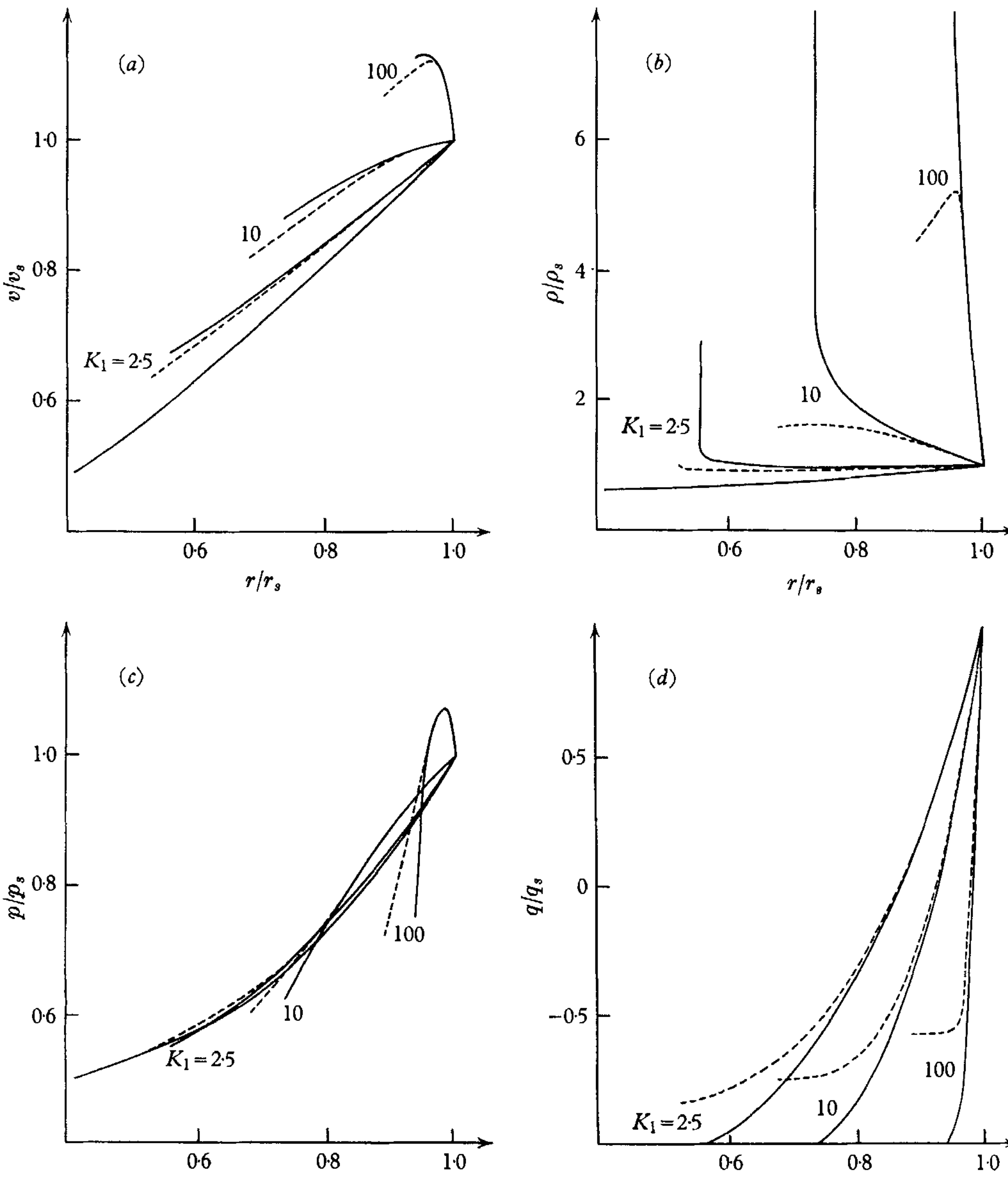

FIGURE 4. (*a*) Velocity distribution. (*b*) Density distribution. (*c*) Pressure distribution. (*d*) Radiative flux distribution. ——, non-radiative; ——, transparent radiative; general radiative: - - -, $\rho_0 = 1{\cdot}29 \times 10^{-5}$. Flow pattern: $\nu = 1, \gamma = \frac{7}{5}, \alpha = 2, \beta = -1$.

The calculations are carried out for a similar range of parameters as before, viz. $\nu = 1, 2, 3$; $\gamma = \frac{7}{5}, \frac{5}{3}$; $\alpha, \beta = (1, 5), (2, -1)$; $\rho_0 = 1{\cdot}29 \times 10^{-4}, 1{\cdot}29 \times 10^{-5}$. The value of Stefan's constant, σ, is taken to be $5{\cdot}735 \times 10^{-5}$ ergs/cm^2 and the value employed for the gas constant, $\mathscr{R}$, is $2{\cdot}882 \times 10^6$ ergs/g degK. For ease of calculation a range of values for the dimensionless constant K_1 is chosen rather than one for the piston speed U_0, and in figures 3–6 is shown the continuous

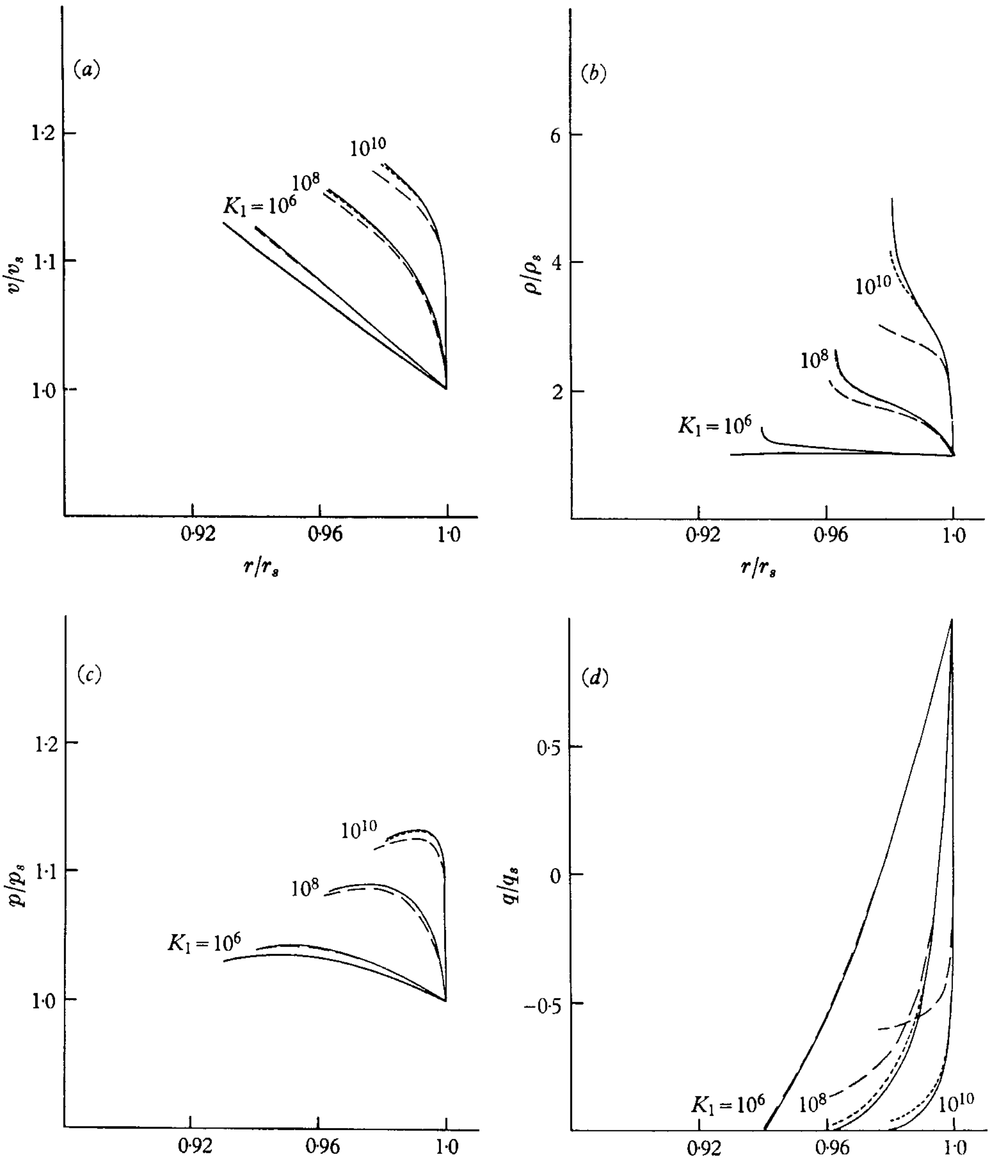

FIGURE 5. (*a*) Velocity distribution. (*b*) Density distribution. (*c*) Pressure distribution. (*d*) Radiative flux distribution. ——, non-radiative; ——, transparent radiative; general radiative: — —, $\rho_0 = 1{\cdot}29 \times 10^{-4}$; - - -, $\rho_0 = 1{\cdot}29 \times 10^{-5}$. Flow patterns: $\nu = 3$, $\gamma = \frac{7}{5}$, $\alpha = 1$, $\beta = 5$.

variation with position of the flow field for the plane and spherical pistons with $\gamma = \frac{7}{5}$. In the transparent limit the results in the plane case agree closely with the earlier computations of Wang cited above, but for the more realistic case with general opacity they differ in several respects.

First of all it is noted that, with increase in the geometrical dimensions, the basic parameters of the flow being unchanged, the value at the piston face of r/r_s increases, which indicates that the piston follows the shock more closely.

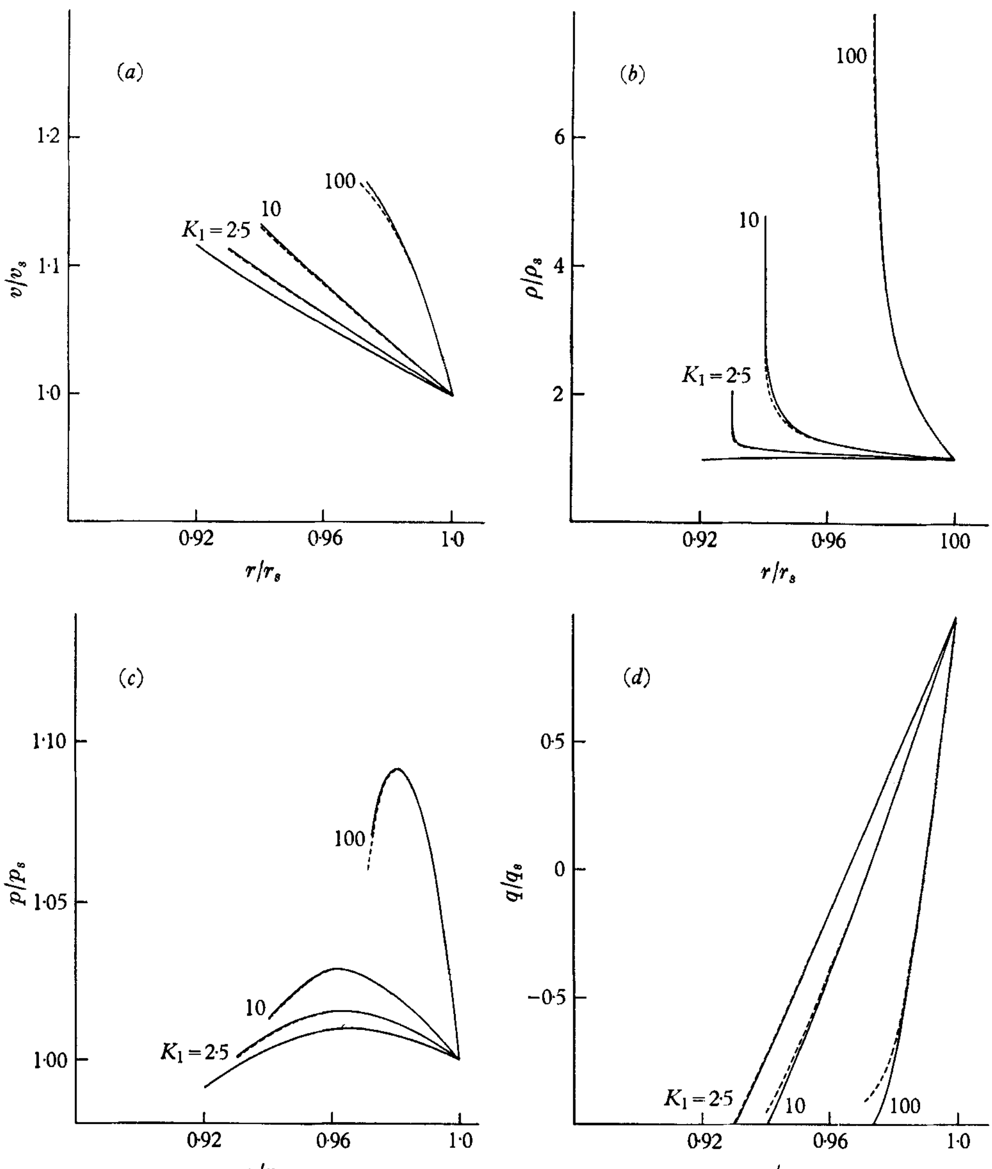

FIGURE 6. (a) Velocity distribution. (b) Density distribution. (c) Pressure distribution. (d) Radiative flux distribution. ———, non-radiative; ———, transparent radiative; general radiative: - - - ., $\rho_0 = 1{\cdot}29 \times 10^{-5}$. Flow pattern: $\nu = 3$, $\gamma = \frac{7}{5}$, $\alpha = 2$, $\beta = -1$.

As the radiative effects become more important through the increase of K_1 and thus of the piston speed, the piston follows even closer behind the shock. However, in the transparent limit these effects are exaggerated and as the density parameter increases so does the optical thickness of the gas and the more exact solution deviates somewhat from the transparent approximation.

The effect of radiation upon the velocity is seen to be quite different for plane and spherical pistons. For the latter its influence is always fairly small and qualitatively is the same as that in the absence of radiation with a rise in the particle speed as one moves back from the shock towards the piston, whilst for the plane case the gas velocity in general falls. However, in this case the effects only remain small at lower temperatures ($\alpha = 1$, $\beta = 5$) and even then become more marked as K_1 increases and a narrow region of steep velocity gradient develops locally behind the shock. At higher temperatures ($\alpha = 2$, $\beta = 1$) the effects of radiation are considerable with the same general properties as before. The use of the transparent limit again exaggerates all radiative effects.

The pressure is little affected by radiation. For the spherical piston the distribution is very similar to that in the non-radiating case with merely a steepening of pressure gradient everywhere but particularly behind the shock. In the plane case, whilst in the absence of radiation the pressure falls as one moves towards the piston, the effect of increasing the piston speed with radiation is to introduce a local increase of pressure behind the shock. In this instance the transparent limit gives a good approximation.

Radiative effects upon the density and temperature distribution are marked. In the non-radiative case the density remains nearly constant, but radiative transfer in the transparent limit causes a rapid increase of density to occur near the piston face, and at lower temperature ($\alpha = 1$, $\beta = 5$) as K_1 increases a similar rapid growth begins to take place behind the shock to form two boundary layers with a slower rate of change in between. On the other hand, at higher temperatures ($\alpha = 2$, $\beta = -1$) no boundary layer develops at the shock. However, the effects are exaggerated in the transparent limit, particularly in the vicinity of the piston and it seems that the boundary layer at the piston face may not develop; in fact the calculations for $\alpha = 2$, $\beta = -1$ suggest that in the plane case the density is actually decreasing there. Similar behaviour applies to the temperature, the distribution of which is easily deduced from the relationship

$$T/T_s = (p/p_s)/(\rho/\rho_s).$$

It appears that it always falls steeply behind the shock wave and continues to decrease as one moves towards the piston.

A few remarks at this stage concerning the effect of the radiative boundary conditions are appropriate. Analysis has shown that, in general, a thermal boundary layer develops at both the shock and the piston face. Since no radiative flux enters the gas from upstream of the shock, a condition formally stated by (16), the consequence is that immediately behind the shock the temperature falls steeply. On the other hand, since the shock is assumed transparent, the radiative flux from downstream passes straight through and is absorbed in the upstream gas. However, in the context of the self-similar model this is lost to the problem, for the strong shock condition demands that all upstream energy is neglected compared with that downstream, whilst in fact it could well be that this contribution counteracts to some extent the temperature drop behind. The question is still an open one and further investigation is necessary to resolve it. A similar situation arises at the piston face, for the assumption of a cool non-radiating piston, stated formally by (17), takes no account of the temperature rise that must result in the piston as radiation is absorbed from the hot gas between it and the shock wave.

The behaviour of the radiative flux is the same for all geometries. As K_1 increases with piston speed the domain of high temperature moves nearer to the shock wave and the flow of radiative energy is generally directed towards the piston except in an increasingly narrow region immediately behind the shock wave. As one may expect, the effect is less marked as the overall temperature of the gas rises, and as the ambient density and hence optical thickness of the gas increases the radiative flux at the piston becomes considerably less than that at the shock. It should be noted that for the plane piston and also for $\gamma = \frac{5}{3}$ (not presented here) the calculations show that the radiative flux computed from the transparent limit is rather inaccurate, for the use of the more exact equations yields a distribution in which the value of q/q_s has a minimum which moves nearer the shock as the piston speed increases.

Finally it is important to note the range of validity of the foregoing analysis, which rests upon the assumption that the precursor shock wave is strong. The condition necessary for this is $c \gg a_0$, where c is the shock speed and a_0 is the acoustic speed in the upstream undisturbed gas. Hence if p_0 is the uniform ambient pressure it follows from (12) and (19) that the results are valid provided that the shock location $r = r_s$ is such that

$$r_s \ll \left\{\frac{\delta^2 \rho_0}{\gamma p_0}\left(\frac{U_0}{\delta\lambda^*}\right)^{2/\delta}\right\}^{5/7\omega}.$$

For instance, using c.g.s. units, if $\rho_0 \simeq 10^{-4}$ and $p_0 \simeq 10^3$ then the solution only holds provided that $r_s \ll 10^8$ when $\alpha = 1$, $\beta = 5$, but $r_s \ll 10^4$ at higher temperatures with $\alpha = 2$, $\beta = -1$.

In addition, it must be observed that, since in the model investigated $\omega > 0$ and $n < 0$, the piston, located initially at the origin, is impulsively set in motion into a gas which is infinitely dense there with a speed which steadily decreases but is at first infinite. Thus, of necessity, the instant $t = 0$ must be excluded from specific physical interpretation although all successive instants of time are acceptable.

REFERENCES

Armstrong, B. H., Sokoloff, J., Nicholls, R. W., Holland, D. H., & Meyerott, R. E. 1961 *J. Quant. Spectrosc. Rad. Transf.* **1**, 143.

Helliwell, J. B. 1966 *Phys. Fluids*, **9**, 1869.

Sedov, L. I. 1959 *Similarity and Dimensional Methods in Mechanics*. New York: Academic Press.

Taylor, G. I. 1946 *Proc. Roy. Soc.* A, **186**, 273.

Wang, K. C. 1964 *J. Fluid Mech.* **20**, 447.